Ethics in Mathematics – Tripos Questions

Analysis I (Sheet 2)

Let $f_n : [0,1] \to [0,1]$ be a continuous function for each $n \in \mathbb{N}$. Find an ugly proof, and a nice proof, of the result that $h_n(x) = \max\{f_1(x), f_2(x), ..., f_n(x)\}$ is continuous on $[0,1]$ for each n. Must the function h defined by $h(x) = \sup\{f_n(x) : n \in \mathbb{N}\}$ be continuous on $[0,1]$?

Solution

The purpose of this question is to prompt the students to think about, and realise, that mathematics is viewed beyond "correct vs incorrect"; we have our own stylistic and professional norms and beliefs on what is "good" mathematics, and what is "bad" mathematics.

Differential Equations (Sheet 4)

The evolution of an infectious disease in a population can be modelled by

$$\dot{U} = U(1 - (U + I)) - \beta UI$$
$$\dot{I} = I(1 - (U + I)) + \beta UI - \delta I$$

where U is the uninfected population, I is the infected population, $\beta > 0$ is the rate of infection, and $\delta > 0$ is the death rate caused by the disease. For $\beta = \frac{3}{4}$ determine the location and stability of the critical points of the above system in the cases:
i) $\delta = \frac{1}{5}$, ii) $\delta = \frac{2}{5}$, iii) $\delta = \frac{3}{5}$.
Thus determine the long-term outcome for the population in each case. Which of these values of δ gives the least total population in the long term? Explain why this occurs. Which do you think is worse: a disease with a mortality rate of $\frac{2}{5}$, or of $\frac{3}{5}$?

Solution

The purpose of this question is to identify what the term worse *means, and in particular, worse* for whom? *A government or hospital may prefer $\delta = \frac{3}{5}$; an individual being infected by the disease would almost certainly prefer $\delta = \frac{2}{5}$. Utilitarianism is great, unless you're on the losing end of it! Mathematicians are good at solving optimisation problems, but are seldom made to question* what *to optimise for.*

Groups (Sheet 4)

The *15-puzzle* consists of 15 small square tiles, numbered 1 to 15, which are mounted in a 4×4 frame in such a way that each tile can slide vertically or horizontally into an adjacent square (if it is not already occupied by another tile), but the tiles cannot be lifted out of the tray. A cash prize was offered for a solution to manoeuvre the tiles from the first to the second of the configurations shown below. Can you do it?
Give such a solution, or show that none exist, and outline the merits and drawbacks of offering a prize for such a puzzle.

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

15	14	13	12
11	10	9	8
7	6	5	4
3	2	1	

Solution

The idea of this question is to prompt the students into asking the question "Should I be using my understanding of mathematics to trick people?" This is a fairly harmless example, but as soon as we consider such puzzles as being sold for profit, then it becomes an issue of exploiting knowledge asymmetry. Of course, any well-educated person would realise that the puzzle is impossible, but not everyone has a mathematics degree; students may not be actively aware of this.

Numbers and Sets (Sheet 1)

Do you agree with the following statement?

> "If you don't do it, then someone else will."

Express the above statement in symbolic notation (using $\implies$, $\impliedby$, $\neg$, etc.); find its contrapositive and its negation, giving each in symbols and in words. Do you think the result makes sense?

Solution

The point of this question is to get the students to show, mathematically, that the paradigm "If you don't do it, someone else will" is quite flawed as a argument or incentive to do something. Hopefully this can equip the students with enough realisation to push back on such an argument, should they ever encounter it in their professional life.

Probability (Sheet 3)

Given a circle of unit radius, what is the probability that a random chord of the circle is longer than the sides of an inscribed equilateral triangle?

Consider the following methods of randomly choosing a chord:

1. Fix a point on the circumference as one end of the chord, and choose an angle $\Theta \sim \text{Unif}(0, \pi)$ to be the angle subtended by the chord at the centre.

2. Fix a radius to intersect the chord at right-angles, and choose a distance from the centre $H \sim \text{Unif}(0, 1)$ to be the midpoint of the chord.

3. Choose a uniformly distributed point in the circle (X, Y) to be the midpoint of the chord (to do this, choose uniform variables $D \sim \text{Unif}(0, 1)$, $\Theta \sim \text{Unif}(0, 2\pi)$, then set $X = \sqrt{D}\cos\Theta$, $Y = \sqrt{D}\sin\Theta$).

For each of these find and sketch the distribution function $F_L(l)$ and the corresponding probability density function $f_L(l)$ for the length L of the chord.
In each case evaluate the probability that L is greater than the side length of an inscribed equilateral triangle.

Reflecting on what you have calculated, what is wrong with the initial question here?

Solution

This is a crucial observation for students studying probability; how we define "random" determines/changes what we can deduce about random processes. To say "A randomly chosen item from a particular set has a 50% *probability of posessing property X" is a meaningless statement* unless *it is fully qualified with* how *that item was "chosen randomly". This is one of the ways mathematics can, and is, used to dupe the public, especially in when probability theory and statistics are involved.*

Vectors and Matrices (Sheet 3)

1. A square matrix with entries in $\mathbb{R}$ is said to be *column-stochastic* if all of its entries are nonnegative and the entries in each column sum to one. Show that every column-stochastic matrix has 1 as an eigenvalue.

2. The Google PageRank algorithm (simplified) works as follows: Each webpage w_i on the web is assigned a value $v_i \geq 0$ such that, if L_i is the set of pages that link to w_i, and n_i is the number of outgoing links from page w_i, then

$$v_i = \sum_{w_j \in L_i} \frac{v_j}{n_j}$$

   That is, each page w_i "donates" $\frac{1}{n_i}$ of its value v_i, uniformly, to each page that it links to. Show that, if every page on the web links to at least one other page, then there is at least one way of assigning values to each webpage that satisfies the above relation.

Solution

This question serves two purposes. First, it shows that a relatively straightfoward piece of first-year mathematics can be used to have a huge *impact on the way the world functions; the Google pagerank algorithm is (or more accurately, was, as it has now been updated) one of the most influential algoritms in the world. It was often referred to as the $100,000,000,000 algorithm (the market value of Google at the time). Second, the question gives a non-unique solution (there may be many eigenvectors with eigenvalue 1); how does Google decide which eigenvector to use? And how does this decision introduce bias and impartiality into the search results; just because a computer does it, doesn't mean it is impartial!*

Vector Calculus (Sheet 2)

Blood flowing in an artery is modelled as an axisymmetric flow in a cylinder of length l and radius R with axis parallel to $\hat{\mathbf{z}}$. The velocity is $\mathbf{v}(r, \theta, z) = v(r)\hat{\mathbf{z}}$, and satisfies

$$\nabla^2 v = -\frac{p}{\eta l} \ , \quad \text{with boundary conditions} \quad \left.\frac{dv}{dr}\right|_{r=0} = 0 \ , \quad v(R) = 0 \ ,$$

where p is the pressure difference between the ends of the artery, and η is the viscosity of blood. Find $v(r)$, and calculate the volumetric flow rate

$$Q = \iint_A \mathbf{v} \cdot \hat{\mathbf{z}} \, dA$$

through a cross section A of constant z. If a build-up of cholesterol reduces the radius R by a factor of $\frac{1}{5}$, by what factor does the pressure increase?
If a person's blood pressure is 50% higher than the "normal" level then they are said to be in a *hypertensive crisis* and require immediate medical attention. Compared to the radius of the artery, how thick would a layer cholesterol need to be to trigger this?

Solution

This is a modelling question, where the mathematics of model itself is quite straightfoward, but the implications for the thing being modeled (in this instance, the health of a human) are incredibly serious. This is a real model, which describes the real health of real people. Sometimes it is easy for mathematicians to become detached from the consequences of their work, through the abstract nature of mathematics. This question is designed to help students realise that their work can have real consequences and effects, in a way they can all understand and relate to.